The double integrals will always have $\xi,\zeta \leq fp \ll p$, so Eq. (13) simplifies to

$$F \cong [\ell n s - \ell n(1-\gamma)]M(f,\theta,\theta_{max}) \quad (14)$$

where

$$M = 4f[1 - \ell n(f/\theta_{max})], \quad \text{for } \theta = 0$$
$$= 2f[1 - \ell n(f/\theta_{max})], \quad \text{for } \theta = \theta_{max}$$

Equation (14) justifies the linear dependence of F on log(s) shown in Fig. 2.

Figure 3 is a graph of F against θ for increasing values of s; as s increases, the difference between F at $\theta = 0$ and F at $\theta = \theta_{max}$ also increases. This behavior matches our physical intuition of what is going on inside the plume. Function F is a measure of the difference between a particle's actual temperature and what its temperature would be if it had traveled the same distance without absorbing any of its neighbors' radiation. Obviously, particles traveling down the central axis of the plume at $\theta = 0$ will absorb more radiation—and thus cool off more slowly—than will particles traveling at the plume edge where $\theta = \theta_{max}$. Thus, F will increase with s, and increase more rapidly with s the smaller the value of θ.

Conclusion

Undoubtedly the rocket plume model presented here is in many respects unrealistic; in real plumes the hot particles are not all the same size and do not all travel at the same velocity, and the particle density never drops abruptly to zero at $\theta = \theta_{max}$, as shown in Fig. 1. Moreover, the geometric optics approximation used in Eq. (2) is not always appropriate, especially when the particles are so cool that most of their radiative energy occurs at wavelengths the same size as—or larger than—the particle radius. Nevertheless, the model does reveal two important aspects of the radiant energy exchange between hot particles in a vacuum. First, the size of the parameter $\beta = \varepsilon k R^2 \Psi/12$ indicates when the interparticle radiant energy exchange is important and when it is not. Second, inside the rocket plume, only neighboring particles will exchange significant amounts of radiant energy, and this leads to a linear dependence of τ_B and F on log(s). Both the β parameter and the nearest neighbor energy exchange will have their analogs in any set of equations used to describe the behavior of hot particles traveling in a vacuum. The simplified model developed above should provide useful guidelines for the construction of those more complex plume models required for a realistic simulation of a solid fuel rocket.

Acknowledgment

This work was sponsored by the Department of the Air Force.

References

[1]Pearce, B. E., "An Approximate Distribution of Particle Mass Flux in a High Altitude Solid Propellant Rocket Plume," *AIAA Journal*, Vol. 12, May 1974, pp. 718–719.

[2]Rothwell, W. S. and Schick, H. L., "Emission Spectra from a Stream of Cooling Particles," *Journal of Spacecraft and Rockets*, Vol. 11, Aug. 1974, pp. 597–598.

[3]Worster, B. W., "Particulate Infrared Radiation in Aluminized Solid-Fuel Rocket Plumes," *Journal of Spacecraft and Rockets*, Vol. 11, April 1974, pp. 260–262.

[4]Fontenot, J. E., "Thermal Radiation from Solid Rocket Plumes at High Altitude," *AIAA Journal*, Vol. 3, May 1965, pp. 970–972.

[5]Weast, R. C., ed., *Handbook of Chemistry and Physics*, 51st ed., The Chemical Rubber Co., Cleveland, OH, 1971, pp. A-160–A-218.

Multiburst Cloud Rise: Theory and Experiment

Alan W. Zimmerman*
TRW Defense Systems Group,
Redondo Beach, California

Nomenclature

a = cloud aspect ratio $(2b/\Delta h)$
b = cloud horizontal radius
D = diameter
H = stabilization altitude of cloud
Δh = cloud thickness
K = diffusivity
N = number of bursts
s = burst separation
S = s/D_{FB}
t = time
W = nuclear weapon yield
x = altitude
α = entrainment constant
ν = kinematic viscosity

Subscripts

ES = extended source
FB = fireball
MB = multiburst
O = initial value
PS = point source
SB = single burst

Introduction

THE dimensions and stabilization altitude of the cloud produced by a near-surface nuclear burst are of interest to the weapons systems designer. The gross motion of the cloud, except at very early times, is like that of a buoyant thermal, and solutions[1-4] are available for a single burst. Morton et al.[1] obtained a closed-form solution for a uniformly and stably stratified fluid. This solution was extended in Ref. 5 for the limiting case of many bursts that are closely spaced and nearly simultaneous. It is the purpose of this paper to compare the predictions of this multiburst model with experimental data and with results from hydrocode calculations.

Very limited experimental data are available for multiburst cloud rise. A few two-dimensional hydrocode calculations are also available where the initial source is smeared out in rings or a sheet. These are treated here as "data." The predictions of single cloud rise by inviscid hydrocodes have been found to agree very well with experimental data, even though the dominant mechanism controlling cloud rise and growth is turbulent mixing with its surroundings. A recent survey[6] for the Defense Nuclear Agency of all experiments, calculations, and multiburst models identified those experiments and calculations utilized here for comparison with the model of Ref. 5.

Received April 13, 1987; revision received Sept. 15, 1987.

*Section Head, Member AIAA

Analysis

The solution of Morton et al.[1] is based on Taylor's[7] simple assumption that the inflow (entrainment) into a thermal is proportional to the mean rise velocity of the thermal. A second basic assumption is that the flow is incompressible. Then the equations representing conservation of mass, momentum, and buoyancy (heat) can be solved to give average properties (velocity, altitude, buoyancy, and horizontal extent) of the thermal.

This single-burst model agrees with data for both tests in the laboratory and nuclear bursts in the atmosphere. The cloud from a nuclear burst rapidly cools to near ambient temperature, and most of the rise is incompressible flow phenomena. Many details such as the physics of entrained particulate and water vapor are ignored, but very good values of rise velocity and stabilization altitude are obtained.

Reference 5 extends the analysis from a point source to a disk source, which would be appropriate to many closely spaced and simultaneous bursts. The reader is referred to Ref. 5 for the conservation equations and solutions for both the point and extended sources.

Less restrictive assumptions are possible at the cost of greater complexity. Escudier and Maxworthy[3] allow arbitrary density differences and introduce a virtual mass in the momentum equation. The virtual mass leads to the correct limiting behavior as the initial cloud density goes to zero.

The incompressibility assumption common to all the quoted results may be violated if the spacing between bursts is sufficiently small. Supersonic cloud rise is predicted for closely spaced bursts[8] when dust/air heat transfer is ignored. Vaporization of dust will tend to limit the cloud temperature and its rise velocity, but the incompressible solution is probably not valid for all spacings.

For a single burst and N closely spaced bursts in the same atmosphere, the following ratios can be formed from the solution in Ref. 5:

$$x_{ES}/x_{PS} = \left(\frac{3}{a}\frac{\alpha_{SB}}{\alpha_{MB}}\right)^{3/4} N^{1/4} \tag{1}$$

$$b_{ES}/b_{PS} = \left(\frac{a}{3}\frac{\alpha_{MB}}{\alpha_{SB}}\right)^{1/4} N^{1/4} \tag{2}$$

For a compact array of bursts (as opposed to a strip) with a spacing s between bursts and a single fireball diameter D_{FB} prior to liftoff, the initial diameter and thickness of the disk are

$$2b_o \cong D_{FB} + \sqrt{N}\, s \tag{3}$$

$$\Delta h = \tfrac{1}{2} D_{FB} \tag{4}$$

Then with $S = s/D_{FB}$,

$$a = \frac{2b_o}{\Delta h_o} = 2(1 + \sqrt{N}\, S) \tag{5}$$

and the limit as $N \to \infty$

$$\frac{H_{MB}}{H_{SB}} = x_{ES}/x_{PS} = \left[\frac{3}{2S}\frac{\alpha_{SB}}{\alpha_{MB}}\right]^{3/4} N^{-1/8} \tag{6}$$

$$\frac{D_{MB}}{D_{SB}} = b_{ES}/b_{PS} = \left[\frac{2S}{3}\frac{\alpha_{MB}}{\alpha_{SB}}\right]^{1/4} N^{3/8} \tag{7}$$

Equation (6) is the same as Eq. (7) in Ref. 5. Note that Eq. (6) and (7) are not correct for S much larger than 1. If the spacing is large, the clouds will be independent, and the term $\sqrt{N}\, S$ in Eq. (3) is not appropriate.

Comparison with Data

There are two series of tests, both sponsored by the Defense Nuclear Agency, where late time-cloud dimensions are reported and the cloud motion is primarily due to buoyancy and turbulent mixing rather than the wind.[6] These are the MISERS BLUFF HE experiments[9,10] (6 and 24 sources) and a series of laboratory scale tests[4,11] (2–7 sources) in a water tank filled with stratified salt water. Hydrocode calculations with 64[12] and 60[13] sources are also reported. These are two-dimensional calculations with the sources approximated initially by a disc or concentric rings.

Most of the tests and calculations are for spacings S near 1, and the data are compared with Eqs. (6) and (7) with $S = 1$. An observation not understood is that experimental values of x_{ES}/x_{PS} are significantly greater than one in air but not in water. This is discussed in the next section. For the purpose of fitting the data, it is found that the use of $\alpha_{MB}/\alpha_{SB} = 0.6$ in air and 1.0 in water gives good agreement.

Note that N is not large for many of the experiments, whereas the theory is valid for large N. The data is used anyway because there is so little.

There are also small changes in the atmospheric density gradient between the different field tests. These are ignored because the dependence is weak (see Ref. 5).

Essential characteristics and cloud dimensions for the tests and hydrocode calculations are presented in Table 1. In all cases, the bursts are surface bursts detonated simultaneously. The lateral separation is the same between nearest neighbors. If several tests are listed on one line, the dimensions given are averages for all the tests noted. The actual separations for the hydrocode calculations and the MISERS BLUFF event were scaled by the cube-root of the yield to a reference yield and then divided by the fireball diameter (for the reference yield) prior to liftoff to give S. The reference yield is the chemical energy in 10^6 tons of TNT. Yield was not computed for the water-tank experiments. The nondimensional spacing S is just the actual spacing divided by the initial bubble diameter (see the footnote to Table 1). Cloud measurements are given at stabilization or at the latest time available if measurements were not continued to stabilization. The dimension H is the height of the cloud top.

A final comment is that H_{SB} for the second set of water-tank experiments is greater than for the first set. As shown by Gorev et al.,[14] the entrainment coefficient is sensitive to the disturbance created in releasing the bubble. A small change was made to the apparatus, resulting in a cleaner release. This is not of concern here because each test series included single-burst experiments, and it is the ratios of multiburst to single-burst dimensions that are of interest. Table 2 presents a comparison of the data with the model described above, viz.,

$$\frac{H_{MB}}{H_{SB}} = \left(\frac{3}{2}\frac{\alpha_{SB}}{\alpha_{MB}}\right)^{3/4} N^{-1/8}$$

$$\frac{D_{MB}}{D_{SB}} = \left(\frac{2}{3}\frac{\alpha_{MB}}{\alpha_{SB}}\right)^{1/4} N^{3/8}$$

$$\frac{\alpha_{MB}}{\alpha_{SB}} = \begin{cases} 0.6 \text{ air} \\ 1.0 \text{ water} \end{cases} \tag{8}$$

Discussion

The predictions of the multiburst model [Eq.(8)] are perhaps better than could be expected considering the approximations. Most of the errors are within the experimental uncertainty.

As discussed in Ref. 5, it is expected that entrainment would be less efficient and the entrainment coefficient smaller for a multiburst event than for a single burst. This is true of experiments in air but not in water. The tests in water differ from nuclear bursts in the scale of the event (not expected to be crucial), in the medium (water rather than air), and in the

Table 1 Multiburst experiments and calculations

Test	W	Number of Bursts	Configuration	S	t	H_{MB}	H_{SB}	D_{MB}	D_{SB}
SHELL OIL calculation[12a]									
	5 MT	64	Rings	0.740	Stable	40 km	26.7 km[b]	130 km	—
MISERS FLUFF[9,10c]									
MBI-1,3,5	1000 lb	1	—	—	100 s	—	1244 ft	—	667 ft
MBI-4	1000 lb	6	Hexagon	1.07	100 s	2500 ft	1244 ft	1210 ft	667 ft
MBI-8	1000 lb	24	Hexagons	1.07	100 s	2480 ft	1244 ft	1890 ft	667 ft
MBII-1	118 tons	1	—	—	600 s		2.5 km		1.9 km
MBII-2	118 tons	6	Hexagon	0.809	600 s	3.7 km	2.5 km	2.84 km	1.9 km
HULL calculations[13]									
	1 MT	60	Rings	1.00	450 s	24.4 km	17.8 km	66.2 km	—
	1 MT	60	Sheet	1.00	450 s	21.0 km	17.8 km	76.0 km	—
Water tank experiments[4]									
1.1,2,3,4	[d]	1	—	—	10.4 s		15.8 in.		12.4 in.
2.13,14,15	[d]	2	Side by side	1.2	10.5 s	16.6 in.	15.8 in.	18.0 in.	12.4 in.
2.2,3,7	[d]	2	" " "	1.3	10.7 s	17.4 in.	15.8 in.	16.4 in.	12.4 in.
2.10,11,12	[d]	2	" " "	2.0	10.5 s	19.0 in.	15.8 in.	17.4 in.	12.4 in.
2.4,5	[d]	2	" " "	3.0	10.6 s	17.2 in.	15.8 in.	18.2 in.	12.4 in.
2.8,9	[d]	2	" " "	6.0	10.3 s	19.0 in.	15.8 in.	e	12.4 in.
3.2,3,4	[d]	3	Triangle	1.2	10.4 s	18.8 in.	15.8 in.	18.4 in.	12.4 in.
TRW water tank experiments[11]									
1.1,2,3	[d]	1	—	—	10 s		17.45 in.		12.94 in.
5.1,3	[d]	5	Square + center	1.2	10 s	19.27 in.	17.45 in.	17.44 in.	12.94 in.
6.1,2,3	[d]	6	Hexagon	1.2	10 s	19.37 in.	17.45 in.	20.86 in.	12.94 in.
6.4,5	[d]	6	Hexagon	2.0	10 s	18.13 in.	17.45 in.	21.26 in.	12.94 in.
6.6,7	[d]	6	Hexagon	3.0	10 s	18.77 in.	17.45 in.	23.57 in.	12.94 in.
7.1,2	[d]	7	Hexagon + center	1.2	10 s	19.55 in.	17.45 in.	20.29 in.	12.94 in.

[a]Secondary reference. Calculations not found. [b]Comparable SB calculation unavailable. H_{SB} from Ref. 1 ($\alpha = 0.25$). Reference 1 prediction of D_{SB} unreasonable. [c]MBI dimensions from Ref. 9 MBII dimensions from Ref. 10 [d]Density gradient was chosen so that single cloud dimensions are equal to those for a 1-MI cloud when the initial hemisphere diameter is scaled up to the diameter of a 1-MI fireball. [e]Not merged at stabilization.

Table 2 Model predictions of cloud dimensions vs values from experiments or hydrocode calculations

Test	Number of bursts	S	H_{MB}/H_{SB} Predicted	H_{MB}/H_{SB} Exp/Calc.	Error in prediction[a]	D_{MB}/D_{SB} Predicted	D_{MB}/D_{SB} Exp/Calc.	Error in prediction[a]
SHELL OIL	64	0.740	1.18	1.50	0.213	3.78	Not available	Not available
MISERS BLUFF								
I-4	6	1.07	1.59	2.01	0.209	1.56	1.81	0.138
I-8	24	1.07	1.34	1.99	0.327	2.62	2.83	0.0742
II-2	6	0.809	1.59	1.48	0.0743	1.56	1.50	0.04
HULL calculation								
Rings	60	1.00	1.19	1.37	0.131	3.69	Not available	Not available
Sheet	60	1.00	1.19	1.18	0.0085	3.69	Not available	Not available
Water tank (Part 1)								
2.13,14,15	2	1.2	1.24	1.05	0.181	1.17	1.45	0.193
2.2,3.7	2	1.3	1.24	1.10	0.127	1.17	1.32	0.114
2.10,11,12	2	2.0	1.24	1.20	0.0333	1.17	1.40	0.164
2.4,5,6	2	3.0	1.24	1.09	0.138	1.17	1.47	0.204
2.8,9	2	6.0	1.24	1.20	0.0333	1.17	Not available	Not available
3.2,3,4	3	1.2	1.18	1.19	0.00840	1.37	1.48	0.0743
Water tank (Part 2)								
5.1,3	5	1.2	1.11	1.10	0.009	1.65	1.35	0.222
6.2,3	6	1.2	1.08	1.11	0.027	1.77	1.61	0.0994
6.4,5	6	2.0	1.08	1.04	0.0385	1.77	1.64	0.0793
6.6,7	6	3.0	1.08	1.08	0	1.77	1.82	0.0275
7.1,2	7	1.2	1.06	1.12	0.0536	1.88	1.57	0.198

[a] $\left| \dfrac{\text{Predicted dimension} - \text{Exp/hydrocode value}}{\text{Exp/hydrocode value}} \right|$

absence of a shock. Also, the density is changed by the addition of salt rather than by heat addition.

Batchelor and Townsend[15] state that the interaction of turbulent and molecular diffusion is often important. Molecular diffusion acts to smooth gradients and makes mixing less efficient. Furthermore, it is stated that this interaction cannot be ignored if the ratio of the diffusivity (for marked fluid into unmarked fluid) to the kinematic viscosity is of order unity or greater. The ratios of interest are

$$\text{Diffusion of heat in air } K/_{v} = 0\ (1)$$

$$\text{Diffusion of salt in water } K/_{v} = 0\ (10^{-3})$$

This explanation is at least consistent with the test results since a lower entrainment coefficient for tests in air means that mixing is less efficient. It is somewhat surprising that this difference between air and water has been manifested here and apparently not in single-burst tests. Such a difference could be masked by the dependence of the absolute value of α on cloud formation conditions and by other experimental uncertainties. It is hoped that further investigation of the entrainment process will clarify this difference.

In summary, it is shown that multiburst cloud rise dimensions can be predicted with reasonable accuracy using a closed-form solution of the conservation equations. An empirical parameter (entrainment coefficient) is required, and this parameter is found to depend on the fluid medium in multiburst experiments.

Acknowledgment

This work was supported by the Defense Nuclear Agency under Contract DNA001-84-0107.

References

[1]Morton, B. R., Taylor, G. I., and Turner, J. S., "Turbulent Gravitational Convection from Maintained and Instantaneous Sources," *Proceeding of the Royal Society*, Vol. 234, Jan. 1956, pp. 1-23.

[2]Turner, J. S., "Buoyancy Effects in Fluids," Chap. 6, Cambridge University Press, Cambridge, MA, 1973.

[3]Escudier, M. P. and Maxworthy, T., "On the Motion of Turbulent Thermals," *Journal of Fluid Mechanics*, Vol. 61, 1973, pp. 541-552.

[4]Mazzola, T. A., "The Interaction of Multiple Buoyant Clouds," Master's Thesis, University of Southern California, May 1985; also Rept. RDA-TR-135611-001, R&D Associates, Jan. 1986.

[5]Zimmerman, A. W., "Multiburst Cloud Rise," *AIAA Journal*, Vol. 16, June 1978, pp. 619-621.

[6]Zimmerman, A. W., "Multiburst Cloud Model: Laterally Separated Simultaneous Bursts," *TRW Electronics and Defense Sector*, H325.10.AWZ.86-20, March 1986.

[7]Taylor, G. I., "Dynamics of a Mass of Hot Gas Rising in Air," U. S. Atomic Energy Commission, MDDC-919, 1945.

[8]McKinley, T. K., "Dust Environments for Flight Vehicle Dispersion and Fratricide Analysis, Data Book IA: 500 MT Spike, Surface Burst Ejecta," *Systems Science and Software*, Oct. 1983.

[9]Wisotki, J., "MISERS BLUFF I and II Technical Photography," Denver Research Institute, DNA POR 6983, Oct. 1980.

[10]Thomas, C. R. and Cockayne, J. E., "MISERS BLUFF II Cloud Sampling Program," Science Applications, Inc., DNA 5189F, Dec. 1979.

[11]Colbert, R. G. and Zimmerman, A. W., "Multiple Burst Experiment in a Water Tank," *TRW Electronics and Defense Sector*, DNA-TR-84-411, Jan. 1985.

[12]Adelman, F. L., "Cloud Rise in a Dense Nuclear Attack," System Planning Corp., DNA 4862F, Aug. 1978.

[13]Filipelli, K. J., "Theoretical Calculations of Mushroom Clouds from Multiburst Spike Attacks," AFWL-TR-80-32, Aug. 1980.

[14]Gorev, V. A., Gusev, P. A., and Troshin, Y. K., "Effect of Formation Conditions on the Motion of a Cloud Rising Upward Under the Action of the Force of Buoyance," *Fluid Dynamics*, June 1977, pp. 770-772, translation from *Izu. Akad. Nauk SSSR, Mekh. Zhidk*, Vol. 11, Sept.-Oct. 1976.

[15]Batchelor, G. K. and Townsend, A. A., "Turbulent Diffusion," *Surveys in Mechanics*, Cambridge University Press, Cambridge, MA, 1956, pp. 352-399.

Source Expansion Solutions for Radiative Transfer in Slab, Spherical, and Cylindrical Geometries

A. Kisomi* and W. H. Sutton†
University of Oklahoma, Norman, Oklahoma

Introduction

RADIATIVE transfer in curvilinear geometries has received relatively little attention. For an isotropically scattering slab with uniform emission, Sutton and Ozisik[1] presented exact exit flux results both for the singular eigenfunction expansion method (separation of variables) and for the Fourier transform method. The analysis of radiative transfer through a finite, spherically symmetric, and uniformly generating medium with exact solution has been reported by Heaslet and Warming.[2] Analytical and numerical methods of predicting radiative transfer were also developed for a homogeneous medium in a circularly symmetric, cylindrical region by Heaslet and Warming.[3]

The analysis reported here is concerned with three geometries common to engineering applications: plane-parallel slab, spherical, and cylindrical media. The primary objective is to formulate and obtain closed-form results, where possible, for the heat flux and radiation intensity. The essential step of the analysis is to reduce the equations to a truncated finite-linear system of algebraic equations by the aid of a Legendre polynomial expansion for the source function. Here we use only the first term of expansion along with the source function definition to illustrate accuracy and computational speed compatible with combined-mode heat-transfer requirements.

Formulation

The radiative transfer equation for an arbitrary region is given by Mitsis[4]

$$\frac{\mathrm{d}I(\boldsymbol{r},\hat{\Omega})}{\mathrm{d}s} + I(\boldsymbol{r},\hat{\Omega}) = S(\boldsymbol{r}) \tag{1}$$

where I is the spectral radiation intensity, Ω is the beam solid angle, $\boldsymbol{r}$ and s are optical variables, and the source function $S(\boldsymbol{r})$ is defined as

$$S(\boldsymbol{r}) = (1-\omega)I_b[\boldsymbol{r}] + \frac{\omega}{4\pi}\int_{4\pi} I(\boldsymbol{r},\hat{\Omega})\,\mathrm{d}\hat{\Omega} \tag{2}$$

In the preceding equation, $I_b[\boldsymbol{r}]$ represents an emission source and ω is the spectral scattering albedo.

Equation (1) can be expressed in terms of the local coordinates in the beam directional $\hat{\Omega}$, where $\boldsymbol{r} = \boldsymbol{r}' - s\hat{\Omega}$, and then solve it formally to get the radiant intensity,[4] where $s_0 = |\boldsymbol{r}_0 - \boldsymbol{r}|$ is the distance from $\boldsymbol{r}$ to the surface point $\boldsymbol{r}_0$, and $I(\boldsymbol{r} - s_0\hat{\Omega},\hat{\Omega})$ represents the surface intensity. For simplicity in the current work, a free boundary (meaning a transparent boundary with no external emission) is assumed. This

Presented as Paper 85-0946 at the AIAA 20th Thermophysics Conference, Williamsburg, VA, June 19-21, 1985; received Jan. 20, 1986; revision received Aug. 4, 1987.

*Graduate Student, School of Aerospace, Mechanical, and Nuclear Engineering. Member AIAA.

†Assistant Professor, School of Aerospace, Mechanical, and Nuclear Engineering. Member AIAA.